

Multivalued generalizations of the Frankl–Pach Theorem

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Abstract

In [13] P. Frankl and J. Pach proved the following uniform version of Sauer’s Lemma.

Let n, d, s be natural numbers such that $d \leq n$, $s + 1 \leq n/2$. Let $\mathcal{F} \subseteq \binom{[n]}{d}$ be an arbitrary d -uniform set system such that \mathcal{F} does not shatter an $s + 1$ -element set, then

$$|\mathcal{F}| \leq \binom{n}{s}.$$

We prove here two generalizations of the above theorem to n -tuple systems. To obtain these results, we use Gröbner basis methods, and describe the standard monomials of Hamming spheres.

Keywords. Gröbner basis, standard monomial, uniform family, shattered set, n -tuple system.

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1 Introduction

Let $[n]$ stand for the set $\{1, 2, \dots, n\}$. The family of all subsets of $[n]$ is denoted by $2^{[n]}$. For an integer $0 \leq d \leq n$ we denote by $\binom{[n]}{d}$ the family of all d element subsets of $[n]$, and by $\binom{[n]}{\leq d} = \binom{[n]}{0} \cup \dots \cup \binom{[n]}{d}$ the family of subsets of size at most d .

Let $n > 0$, $\mathcal{F} \subseteq 2^{[n]}$ be a family of subsets of $[n]$, and S be a subset of $[n]$. We say that \mathcal{F} *shatters* S if

$$\{F \cap S : F \in \mathcal{F}\} = 2^S. \quad (1)$$

Define

$$\text{sh}(\mathcal{F}) = \{S \subseteq [n] : \mathcal{F} \text{ shatters } S\}. \quad (2)$$

The following result was proved by Sauer, [22], and independently by Vapnik and Chervonenkis [26], and Perles and Shelah [23]:

Theorem 1.1 *Suppose that $0 \leq s \leq n - 1$ and let $\mathcal{F} \subseteq 2^{[n]}$ be an arbitrary set family with no shattered set of size $s + 1$. Then*

$$|\mathcal{F}| \leq \sum_{i=0}^s \binom{n}{i}.$$

Karpovsky and Milman in [18] gave a generalization of Sauer's result for tuple systems. Next we explain this multivalued generalization. Throughout the paper $q \geq 2$ is an integer. Let (q) stand for the set $\{0, 1, \dots, q - 1\}$ and denote by \mathbf{v}_F the characteristic vector of the set $F \subseteq [n]$. Clearly we have $\mathbf{v}_F \in (2)^n$.

Subsets $\mathcal{V} \subseteq (q)^n$ will be called *tuple systems*¹. Note that an element \mathbf{v} of a tuple system \mathcal{V} can also be viewed as a function from $[n]$ to (q) . With this in mind, we say that the tuple system \mathcal{V} shatters a set $S \subseteq [n]$, if

$$\{\mathbf{v} \upharpoonright_S : \mathbf{v} \in \mathcal{V}\}$$

is the set of all functions from S to (q) ; here $\mathbf{v} \upharpoonright_S$ denotes the restriction of the function \mathbf{v} to the set S . This extends the binary notion of shattering introduced in (1). In fact, consider

$$\text{Sh}(\mathcal{V}) := \{S \subseteq [n] : \mathcal{V} \text{ shatters } S\}, \quad (3)$$

¹They are also called *sets of vectors* in the literature.

the set of the shattered sets of the tuple system \mathcal{V} .

Clearly $\text{Sh}(\mathcal{V}) \subseteq 2^{[n]}$. Moreover, if $\mathcal{F} \subseteq 2^{[n]}$ is a set system, then

$$\text{sh}(\mathcal{F}) = \text{Sh}(\{\mathbf{v}_F \in 2^{(n)} : F \in \mathcal{F}\}).$$

The following result was proved by Karpovsky and Milman in [18, Theorem 2] (see also Alon [1, Corollary 1], Steel [22, Theorem 2.1] and Anstee [3, Theorem 1.3]).

Theorem 1.2 *Let $0 \leq s \leq n - 1$ be an integer and let $\mathcal{V} \subseteq (q)^n$ be a tuple system with no shattered set of size $s + 1$. Then*

$$|\mathcal{V}| \leq \sum_{i=0}^s (q-1)^{n-i} \binom{n}{i}.$$

□

The above theorem can be viewed as a natural multivalued generalization of Theorem 1.1.

A set family $\mathcal{F} \subseteq 2^{[n]}$ is called d -uniform, iff $|F| = d$ holds, whenever $F \in \mathcal{F}$. Uniformity can be generalized to tuple systems in two simple ways. First let $0 \leq d \leq (q-1)n$. A tuple system $\mathcal{V} \subseteq (q)^n$ is d -uniform iff $\sum_{i=1}^n v_i = d$ holds for every $(v_1, \dots, v_n) \in \mathcal{V}$.

Alternatively, let $0 \leq d \leq n$. A tuple system $\mathcal{V} \subseteq (q)^n$ is d -Hamming, if $|\{i \in [n] : v_i \neq 0\}| = d$ for every $(v_1, \dots, v_n) \in \mathcal{V}$.

In [13] P. Frankl and J. Pach proved the following uniform version of Theorem 1.1.

Theorem 1.3 *Let n, d, s be natural numbers such that $d \leq n$, $s + 1 \leq n/2$. Let $\mathcal{F} \subseteq \binom{[n]}{d}$ be an arbitrary d -uniform set system such that \mathcal{F} does not shatter an $s + 1$ -element set, then*

$$|\mathcal{F}| \leq \binom{n}{s}.$$

We would like to extend this result to tuple systems and hence obtain uniform variants of the Karpovsky–Milman theorem. We prove the following two theorems, which specialize to the Frankl–Pach bound in the case $q = 2$.

Theorem 1.4 *Suppose that $0 \leq d \leq n(q-1)$ and $s \leq \frac{n}{2}$. Let \mathcal{V} be an arbitrary d -uniform tuple system with no shattered set of size $s+1$. Then*

$$|\mathcal{V}| \leq \sum_{i=0}^s (q-1)^{n-i} \left(\binom{n}{i} - \binom{n}{i-1} \right).$$

Theorem 1.5 *Suppose that $0 \leq d \leq n$ and $0 \leq d+s \leq n$. Let \mathcal{V} be an arbitrary d -Hamming tuple system with no shattered set of size $s+1$. Then*

$$|\mathcal{V}| \leq \binom{n}{s} \sum_{i=0}^d \binom{n-s}{i} (q-2)^i.$$

The rest of the paper is organized as follows: Section 2 contains our basic results involving Gröbner bases and normal sets. Sections 3 and 4 contain the proofs of Theorems 1.4 and 1.5. The paper ends with some concluding remarks.

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2 Gröbner bases, standard monomials and shattering

Next we fix some notation related to Gröbner bases in polynomial rings, we need later on. The interested reader can find a detailed introduction to this topic in the classic papers by Buchberger [7], [8], [9], and in the textbooks [2], [10], [11].

We shall work over the field of rational numbers \mathbb{Q} and we denote by $R := \mathbb{Q}[x_1, \dots, x_n]$ the polynomial ring in n variables over \mathbb{Q} . We fix a monomial order \prec on R such that $x_n \prec x_{n-1} \prec \dots \prec x_1$ holds. For a nonzero polynomial $f \in R$ we denote by $\text{lm}(f)$ the largest monomial of f with respect to \prec .

Let I be a nonzero ideal of R . Recall, that a finite subset $\mathcal{G} \subseteq I$ is a *Gröbner basis* of I (with respect to \prec) if for every $f \in I$ there exists a $g \in \mathcal{G}$ such that $\text{lm}(g)$ divides $\text{lm}(f)$.

We shall denote by $\text{SM}(I)$ the set of all standard monomials of I with respect to the term-order \prec over \mathbb{Q} . $\text{SM}(I)$ is often called as a *normal set* of

I . $\text{SM}(I)$ is the complement of $\text{LM}(I)$, the set of all leading monomials for I within the set of all monomials of R . It is known that for a nonzero ideal I (the image of) $\text{SM}(I)$ is a basis of the \mathbb{Q} -vector-space R/I .

We denote by $NF(f, \mathcal{G})$ the (unique) normal form of a polynomial $f \in R$ with respect to a Gröbner basis \mathcal{G} .

To study the polynomial functions on a (finite) set of vectors $\mathcal{V} \subseteq \mathbb{Q}^n$, it is convenient to work with the ideal $I(\mathcal{V})$:

$$I(\mathcal{V}) := \{f \in R : f(\mathbf{v}) = 0 \text{ whenever } \mathbf{v} \in \mathcal{V}\}.$$

It is immediate that $\text{SM}(I(\mathcal{V}))$ is downward closed: if $y \in \text{SM}(I(\mathcal{V}))$, y_1, y_2 are monomials from R such that $y = y_1 y_2$ then $y_1 \in \text{SM}(I(\mathcal{V}))$.

An easy interpolation argument shows that any function from \mathcal{V} to \mathbb{Q} is a polynomial. This gives a bijection from \mathcal{V} to $\text{SM}(I(\mathcal{V}))$. We obtain in particular, that

$$|\text{SM}(I(\mathcal{V}))| = |\mathcal{V}|. \quad (4)$$

2.1 Standard monomials and shattering

The following example illustrates some of the notions we have mentioned so far. Also, it will be useful later in the paper.

Example. We describe a Gröbner basis and the standard monomials of the set $(q)^n \subseteq \mathbb{Q}^n$. We introduce the polynomials

$$f_i(x_i) := \prod_{j=0}^{q-1} (x_i - j) \in \mathbb{Q}[x_1, \dots, x_n] \quad (5)$$

for $1 \leq i \leq n$. These polynomials vanish on $(q)^n$, and the leading monomial of $f_i(x_i)$ is x_i^q . These imply that $\text{SM}(I((q)^n))$ is a subset of $\{\mathbf{x}^{\mathbf{v}} : \mathbf{v} \in (q)^n\}$. But this latter set has q^n elements, hence by (4) we have

$$\text{SM}(I((q)^n)) = \{\mathbf{x}^{\mathbf{v}} : \mathbf{v} \in (q)^n\}. \quad (6)$$

This in turn implies that $\mathcal{G} = \{f_1(x_1), \dots, f_n(x_n)\}$ is a Gröbner basis for $I((q)^n)$.

Next we prove a statement, which connects the notion of shattering to the theory of Gröbner bases.

Proposition 2.1 *Let $\mathcal{V} \subseteq (q)^n$ be a set of tuples. If $S = \{i_1, \dots, i_k\} \subseteq [n]$ is a set for which $x_{i_1}^{q-1} \dots x_{i_k}^{q-1} \in \text{SM}(I(\mathcal{V}))$, then $S \in \text{Sh}(\mathcal{V})$.*

Proof. Suppose that $S \notin \text{Sh}(\mathcal{V})$. We show that $x_{i_1}^{q-1} \dots x_{i_k}^{q-1} \notin \text{SM}(I(\mathcal{V}))$. As $S \notin \text{Sh}(\mathcal{V})$, there exists a tuple $\mathbf{w} = (w_1, \dots, w_n) \in (q)^n$ such that $\mathbf{w} \upharpoonright_S \neq \mathbf{v} \upharpoonright_S$ holds for every $\mathbf{v} \in \mathcal{V}$.

Consider now the polynomial

$$g(x_1, \dots, x_n) := \prod_{j \in S} h_j(x_j) \in \mathbb{Q}[x_1, \dots, x_n],$$

where

$$h_j(x_j) := \prod_{i=0, i \neq w_j}^{q-1} (x_j - i) \in \mathbb{Q}[x_j].$$

Then we immediately see that

$$\text{lm}(g) = x_{i_1}^{q-1} \dots x_{i_k}^{q-1}. \quad (7)$$

We claim that $g(\mathbf{v}) = 0$ holds for every $\mathbf{v} \in \mathcal{V}$. Indeed, let $\mathbf{v} = (v_1, \dots, v_n) \in \mathcal{V}$ be an arbitrary tuple. Since $\mathbf{w} \upharpoonright_S \neq \mathbf{v} \upharpoonright_S$, there must exist an index $j \in S$ such that $w_j \neq v_j$. Then

$$h_j(v_j) = \prod_{i=0, i \neq w_j}^{q-1} (v_j - i) = 0, \quad (8)$$

implying that $g(\mathbf{v}) = 0$. We obtained that $g \in I(\mathcal{V})$. This, together with (7) implies that

$$x_{i_1}^{q-1} \dots x_{i_k}^{q-1} = \text{lm}(g) \notin \text{Sm}(I(\mathcal{V})).$$

□

2.2 The blow-up of a set family

Let $\mathbf{v} \in \mathbb{Q}^n$ be an n -tuple, and put

$$\text{supp}(\mathbf{v}) := \{i \in [n] : v_i \neq 0\}.$$

Let $\mathcal{F} \subseteq 2^{[n]}$ be a set system. We define the *blow-up* $\mathcal{F}^q \subseteq (q)^n$ of \mathcal{F} as

$$\mathcal{F}^q := \{\mathbf{v} \in (q)^n : \text{supp}(\mathbf{v}) \in \mathcal{F}\}.$$

Clearly

$$|\mathcal{F}^q| = \sum_{F \in \mathcal{F}} (q-1)^{|F|}.$$

For a subset $J \subseteq [n]$, we consider

$$\mathcal{F}_J := \{F \in \mathcal{F} : J \subseteq F\} \subseteq 2^{[n]}.$$

Let $g(x_1, \dots, x_n) \in \mathbb{Q}[x_1, \dots, x_n]$ be a polynomial. We define $\bar{g}(x_1, \dots, x_n) := g(p(x_1), \dots, p(x_n))$, where $p(x) \in \mathbb{Q}[x]$ is the unique polynomial for which $\deg(p) = q-1$, $p(0) = 0$ and $p(i) = 1$ for each $1 \leq i \leq q-1$. Clearly we have $\text{lm}(\bar{g}) = \text{lm}(g)^{q-1}$.

For a tuple $\mathbf{v} = (v_1, \dots, v_n) \in (q)^n$ we define three subsets $J(\mathbf{v}), Q(\mathbf{v})$ and $Z(\mathbf{v})$ of $[n]$ as follows:

$$J(\mathbf{v}) = \{i \in [n] : 0 < v_i < q-1\}, \quad Q(\mathbf{v}) = \{i \in [n] : v_i = q-1\}, \text{ and}$$

$Z(\mathbf{v}) = \{i \in [n] : v_i = 0\}$. The sets $J(\mathbf{v}), Q(\mathbf{v}), Z(\mathbf{v})$ partition $[n]$. We note also that a set family $\mathcal{F} \subseteq 2^{[n]}$ can be identified with the tuple system

$$\{\mathbf{v}_F : F \in \mathcal{F}\} \subseteq (2)^n.$$

Here \mathbf{v}_F denotes the characteristic vector of a set $F \subseteq [n]$. This way we can speak of Gröbner bases and standard monomials for a set family \mathcal{F} .

The next result, which may be of independent interest, relates the Gröbner bases and normal sets of \mathcal{F}^q to those of the set systems \mathcal{F}_J , $J \subseteq [n]$. It establishes a useful connection of the multivalued case to the sometimes simpler binary case. We recall first that the polynomials f_1, \dots, f_n from (5) form a Gröbner basis of the ideal of $(q)^n$.

For a subset $J \subseteq [n]$ x_J denotes the monomial $x_J := \prod_{j \in J} x_j$. In particular, $x_\emptyset = 1$.

Theorem 2.2 *Let $\mathcal{F} \subseteq [n]$ be a nonempty set family. For $J \subseteq [n]$, let $\mathcal{G}(\mathcal{F}_J)$ denote a fixed Gröbner basis of the ideal $I(V(\mathcal{F}_J))$. Then*

$$\{f_1, \dots, f_n\} \cup (\cup_{J \subseteq [n]} \{x_J \cdot \bar{g} : g \in \mathcal{G}(\mathcal{F}_J)\}) \cup \{x_J : J \subseteq [n], \mathcal{F}_J = \emptyset\} \quad (9)$$

is a Gröbner basis of the ideal $I(\mathcal{F}^q)$. Moreover,

$$\text{SM}(I(\mathcal{F}^q)) = \{\mathbf{x}^{\mathbf{v}} : \mathbf{v} \in (q)^n, \mathcal{F}_{J(\mathbf{v})} \neq \emptyset, \text{ and } x_{Q(\mathbf{v})} \in \text{SM}(I(\mathcal{F}_{J(\mathbf{v})}))\}. \quad (10)$$

Proof. We note first that the polynomials from (9) clearly vanish on \mathcal{F}^q . Let \mathcal{R} denote the right hand side of (10). To establish the Theorem, it suffices to prove that $|\mathcal{R}| = |\mathcal{F}^q|$, and for each $y = \mathbf{x}^{\mathbf{v}} \notin \mathcal{R}$ there exists a polynomial h from the set (9) such that the leading monomial of h divides y .

Indeed, then $y \in \text{LM}(I(\mathcal{F}^q))$. Using also (6) we obtain that $\text{SM}(I(\mathcal{F}^q)) \subseteq \mathcal{R}$. But then $|\mathcal{R}| = |\mathcal{F}^q| = |\text{SM}(I(\mathcal{F}^q))|$ implies that $\text{SM}(I(\mathcal{F}^q)) = \mathcal{R}$ and in turn gives that the union (9) constitutes a Gröbner basis of the ideal $I(\mathcal{F}^q)$.

First we prove that $|\mathcal{R}| = |\mathcal{F}^q|$. For each $J \subseteq [n]$ such that $\mathcal{F}_J \neq \emptyset$ we fix a bijection

$$\phi_J : \{F \in \mathcal{F} : J \subseteq F\} \rightarrow \text{SM}(I(\mathcal{F}_J)).$$

From (4) we see that

$$|\text{SM}(I(\mathcal{F}_J))| = |\mathcal{F}_J| = |\{F \in \mathcal{F} : J \subseteq F\}|,$$

hence such maps exist. Next we show that the following is a disjoint union decomposition of \mathcal{R} :

$$\mathcal{R} = \cup_{F \in \mathcal{F}} \cup_{J \subseteq F} \{\mathbf{x}^{\mathbf{v}} : \mathbf{v} \in (q)^n, J(\mathbf{v}) = J \text{ and } x_{Q(\mathbf{v})} = \phi_J(F)\}. \quad (11)$$

Indeed, a monomial $\mathbf{x}^{\mathbf{v}}$ from the right side belongs to \mathcal{R} , because $\phi_J(F)$ is in $\text{SM}(I(\mathcal{F}_{J(\mathbf{v})}))$. Conversely, if $\mathbf{x}^{\mathbf{v}} \in \mathcal{R}$, then $x_{Q(\mathbf{v})} = \phi_J(F)$ for some $F \in \mathcal{F}$ with $J \subseteq F$, because ϕ_J is surjective.

Let $J \subseteq F \subseteq [n]$ be fixed subsets, with $F \in \mathcal{F}$. Then $\mathcal{F}_J \neq \emptyset$ and we have

$$|\{\mathbf{x}^{\mathbf{v}} \in \mathcal{R} : J(\mathbf{v}) = J \text{ and } x_{Q(\mathbf{v})} = \phi_J(F)\}| = (q-2)^{|J|}. \quad (12)$$

Keeping this in mind, for a fixed $F \in \mathcal{F}$ we have

$$\begin{aligned} & |\cup_{J \subseteq F} \{\mathbf{x}^{\mathbf{v}} : \mathbf{v} \in (q)^n, J(\mathbf{v}) = J \text{ and } x_{Q(\mathbf{v})} = \phi_J(F)\}| = \\ & \sum_{J \subseteq F} |\{\mathbf{x}^{\mathbf{v}} : \mathbf{v} \in (q)^n, J(\mathbf{v}) = J \text{ and } x_{Q(\mathbf{v})} = \phi_J(F)\}| = \\ & \sum_{J \subseteq F} (q-2)^{|J|} = \sum_{i=0}^{|F|} \binom{|F|}{i} (q-2)^i = (q-1)^{|F|}. \end{aligned}$$

Using again that (11) is a disjoint decomposition, we infer that

$$|\mathcal{R}| = \sum_{F \in \mathcal{F}} (q-1)^{|F|} = |\mathcal{F}^q|.$$

Finally, we prove that if $y = \mathbf{x}^{\mathbf{v}} \notin \mathcal{R}$, then $y \in \text{LM}(I(\mathcal{F}^q))$, more precisely, y is divided by the leading monomial of some polynomial h from (9).

If $v_i > q - 1$, then $h = f_i(x_i)$ will do. We can therefore assume, that $\mathbf{v} \in (q)^n$. Now if $\mathcal{F}_{J(\mathbf{v})} = \emptyset$, then $h = x_{J(\mathbf{v})}$ is a good choice. We are left with the case $\mathcal{F}_{J(\mathbf{v})} \neq \emptyset$. Then $\mathbf{x}^{\mathbf{v}} \notin \mathcal{R}$ is possible only if $x_{Q(\mathbf{v})}$ is a leading monomial for the ideal $I(\mathcal{F}_{J(\mathbf{v})})$, hence there exists a $g \in \mathcal{G}(\mathcal{F}_{J(\mathbf{v})})$ whose leading term divides $x_{Q(\mathbf{v})}$. Taking also into consideration that $x_{J(\mathbf{v})}$ and $x_{Q(\mathbf{v})}$ are relatively prime, we obtain that the leading term of $x_{J(\mathbf{v})} \cdot \bar{g}$ divides y . This finishes the proof. \square

3 The proof of Theorem 1.4

Let $0 \leq d \leq (q - 1)n$. We define the *complete d -uniform tuple system* $\mathcal{U}(n, d, q)$ as follows:

$$\mathcal{U}(n, d, q) := \{\mathbf{v} = (v_1, \dots, v_n) \in (q)^n : \sum_{i=1}^n v_i = d\}.$$

The following result of the authors from [17] gives the standard monomials for the ideal of $\mathcal{U}(n, d, 2)$.

Theorem 3.1 *Suppose that $0 \leq d \leq n$, and set $k = \min\{d, n - d\}$. Let \prec be an arbitrary term order with $x_n \prec \dots \prec x_1$. Then the set of standard monomials of $\mathcal{U}(n, d, 2) \subset (2)^n$ is*

$$\{x_U : U = \{u_1 < \dots < u_\ell\}, \text{ where } \ell \leq k \text{ and } u_i \geq 2i \text{ for } 1 \leq i \leq \ell\}.$$

\square

The sets U appearing in the theorem are essentially the *ballot sequences* (see [19] or [21]): the characteristic vector of U , when viewed as a sequence, has at least as many zeros as ones in any initial segment.

We shall use the approach of [17] to obtain an upper bound for the low degree standard monomials of $I(\mathcal{U}(n, d, q))$. First we set

$$\mathcal{B} = \mathcal{B}(n, q) = \{\mathbf{x}^{\mathbf{v}} : \mathbf{v} \in (q)^n, \quad |\{i \leq 2t - 1 : v_i = q - 1\}| \leq t - 1 \text{ for all } t\}.$$

Next we recall the definition of $\mathcal{H}(t)$ from [17], where it was used in the description of the leading monomials for $\mathcal{U}(n, d, 2)$. Let t be a integer, $0 < t \leq n/2$. We define $\mathcal{H}(t)$ as the set of those subsets $H = \{s_1 < s_2 < \dots < s_t\}$ of $[n]$ for which t is the smallest index j with $s_j < 2j$. Thus, the elements of $\mathcal{H}(t)$ are t -subsets of $[n]$. We have $H \in \mathcal{H}(t)$ iff $s_1 \geq 2, \dots, s_{t-1} \geq 2t - 2$ and $s_t < 2t$. For the first few values of t it is easy to give $\mathcal{H}(t)$ explicitly: we have $\mathcal{H}(1) = \{\{1\}\}$, $\mathcal{H}(2) = \{\{2, 3\}\}$, and $\mathcal{H}(3) = \{\{2, 4, 5\}, \{3, 4, 5\}\}$.

Now let $0 < t \leq n/2$, $0 \leq d \leq n$ and $H \in \mathcal{H}(t)$. Put

$$H' = H \cup \{2t, 2t + 1, \dots, n\} \subseteq [n].$$

Let \mathcal{B}^c stand for the set of monomials in R which are not in \mathcal{B} .

Proposition 3.2 *We have $\mathcal{B}^c \subseteq \text{LM}(I(\mathcal{U}(n, d, q)))$.*

Proof. Let $\mathbf{x}^{\mathbf{v}} \in \mathcal{B}^c$, with $\mathbf{v} = (v_1, \dots, v_n)$. If there is an i such that $v_i \geq q$, then the statement is obvious, $\mathbf{x}^{\mathbf{v}}$ is a leading monomial even for $I((q)^n)$. We can therefore assume that $\mathbf{v} \in (q)^n$. We define now the following tuple $\mathbf{w} = (w_1, \dots, w_n) \in (2)^n$:

$$w_i := \begin{cases} 1 & \text{if } v_i = q - 1 \\ 0 & \text{if } v_i < q - 1. \end{cases}$$

Let $F = F_{\mathbf{v}}$ be the unique subset of $[n]$ such that $\mathbf{w} = \mathbf{v}_F$, where \mathbf{v}_F stands for the characteristic vector of the set F . By our assumption on $\mathbf{x}^{\mathbf{v}}$, there exists a positive integer t and a $H \in \mathcal{H}(t)$ such that $H \subseteq F_{\mathbf{v}}$. Then, writing $H = \{h_1 < \dots < h_t\}$, from the definition of $F_{\mathbf{v}}$ we see that $x_{h_1}^{q-1} \dots x_{h_t}^{q-1}$ divides $\mathbf{x}^{\mathbf{v}}$. Thus, it suffices to prove that $x_{h_1}^{q-1} \dots x_{h_{t-1}}^{q-1} x_{h_t} \in \text{LM}(I(\mathcal{U}(n, d, q)))$, because then $\mathbf{x}^{\mathbf{v}} \in \text{LM}(I(\mathcal{U}(n, d, q)))$ holds as well.

Consider the following polynomial:

$$f(x_1, \dots, x_n) := \prod_{i=0}^{(q-1)(t-1)} \left(\sum_{h \in H'} x_h - (d - i) \right) \in \mathbb{Q}[x_1, \dots, x_n].$$

We claim that

$$f \in I(\mathcal{U}(n, d, q)).$$

Indeed, let $\mathbf{u} = (u_1, \dots, u_n) \in \mathcal{U}(n, d, q)$ be an arbitrary tuple. Then

$$\sum_{h \in H'} u_h = \sum_{i=1}^n u_i - \sum_{j \in [n] \setminus H'} u_j = d - \sum_{j \in [n] \setminus H'} u_j. \quad (13)$$

But clearly $|[n] \setminus H'| = t - 1$, therefore

$$0 \leq \sum_{j \in [n] \setminus H'} u_j \leq (t - 1)(q - 1). \quad (14)$$

Equations (13) and (14) imply that

$$d - (t - 1)(q - 1) \leq \sum_{h \in H'} u_h \leq d.$$

This means that there exists an i with $0 \leq i \leq (t - 1)(q - 1)$ such that $d - i = \sum_{h \in H'} u_h$, giving that $f(\mathbf{u}) = 0$.

Let us consider the polynomials $f_i(x_i)$ from (5) for $1 \leq i \leq n$. Clearly $\mathcal{G} = \{f_1(x_1), \dots, f_n(x_n)\} \subseteq I(\mathcal{U}(n, d, q))$. We shall examine the normal form $NF(f, \mathcal{G})$ of f with respect to \mathcal{G} . We have $NF(f, \mathcal{G}) \in \mathbb{Q}[x_1, \dots, x_n]$.

The multinomial theorem gives that $\frac{((t-1)(q-1)+1)!}{((q-1)!)^{t-1}} \neq 0$ is the coefficient of the monomial $y = x_{h_1}^{q-1} \cdots x_{h_{t-1}}^{q-1} x_{h_t}$ in f . This monomial is not affected by the reduction process with respect to \mathcal{G} , as it is not divisible by x_j^q for any j , it is of top degree $(q - 1)(t - 1) + 1$ in f , and because reduction with respect to \mathcal{G} strictly decreases the degree (the leading monomial of f_i is the only top degree monomial of f_i). These imply that y occurs among the monomials of $NF(f, \mathcal{G})$ as well.

In fact, any monomial y' in $NF(f, \mathcal{G})$ has total degree at most $(q - 1)(t - 1) + 1$, it has degree at most $q - 1$ in any of the variables x_j . Moreover, it is composed of the variables x_h , for $h \in H'$. Among these monomials y is obviously the largest one with respect to \prec . This implies that y is the leading monomial of $NF(f, \mathcal{G})$:

$$\text{lm}(NF(f, \mathcal{G})) = x_{h_1}^{q-1} \cdots x_{h_{t-1}}^{q-1} x_{h_t}. \quad (15)$$

Moreover, $f \in I(\mathcal{U}(n, d, q))$ and $\mathcal{G} \subseteq I(\mathcal{U}(n, d, q))$ imply that $NF(f, \mathcal{G}) \in I(\mathcal{U}(n, d, q))$. This fact and (15) show that

$$x_{h_1}^{q-1} \cdots x_{h_{t-1}}^{q-1} x_{h_t} = \text{lm}(NF(f, \mathcal{G})) \in \text{LM}(I(\mathcal{U}(n, d, q))).$$

□

Corollary 3.3 *We have $\text{SM}(I(\mathcal{U}(n, d, q))) \subseteq \mathcal{B}$.*

□

For an integer $0 \leq i \leq n$ we set

$$X_i = X_i(n, q) := \{\mathbf{x}^{\mathbf{u}} : \mathbf{u} = (u_1 \dots, u_n) \in (q)^n \text{ and } |\{j : u_j = q-1\}| = i\},$$

and similarly

$$X_{\leq i} = X_{\leq i}(n, q) := \{\mathbf{x}^{\mathbf{u}} : \mathbf{u} = (u_1 \dots, u_n) \in (q)^n \text{ and } |\{j : u_j = q-1\}| \leq i\}.$$

Proposition 3.4 *Let $\mathcal{V} \subseteq \mathcal{U}(n, d, q)$ be a d -uniform family such that $\text{Sh}(\mathcal{V}) \subseteq \binom{[n]}{\leq s}$. Then*

$$|\mathcal{V}| \leq |\text{SM}(I(\mathcal{U}(n, d, q))) \cap X_{\leq s}|.$$

Proof. Since $\mathcal{V} \subseteq \mathcal{U}(n, d, q)$, we have $\text{SM}(I(\mathcal{V})) \subseteq \text{SM}(I(\mathcal{U}(n, d, q)))$. On the other hand, $\text{Sh}(\mathcal{V}) \subseteq \binom{[n]}{\leq s}$ implies by Proposition 2.1 that $\text{SM}(I(\mathcal{V})) \subseteq X_{\leq s}$.

We obtain that $\text{SM}(I(\mathcal{V})) \subseteq \text{SM}(I(\mathcal{U}(n, d, q))) \cap X_{\leq s}$. The statement now follows, since by (4) we have $|\mathcal{V}| = |\text{SM}(I(\mathcal{V}))|$. \square

Lemma 3.5 *Suppose that $0 \leq i \leq n/2$. Then*

$$|\mathcal{B} \cap X_i| = (q-1)^{n-i} \left(\binom{n}{i} - \binom{n}{i-1} \right).$$

Proof. We set

$$\mathcal{W}(q, i) := \{\mathbf{w} \in (q)^n : \mathbf{x}^{\mathbf{w}} \in \mathcal{B} \cap X_i\}.$$

Obviously we have $|\mathcal{B} \cap X_i| = |\mathcal{W}(q, i)|$. The elements of $\mathcal{W}(q, i)$ are q -ary analogs of ballot sequences: in each initial segment they have at least as many components with value less than $q-1$ as components with value exactly $q-1$; moreover, the total number of components of the latter type is i .

Consider now the following map F from $(q)^n$ to $(2)^n$:

$$F(\mathbf{v})_i := \begin{cases} 1 & \text{if } v_i = q-1 \\ 0 & \text{if } v_i < q-1. \end{cases}$$

We observe that $G := F|_{\mathcal{W}(q, i)} : \mathcal{W}(q, i) \rightarrow \mathcal{W}(2, i)$ is onto, and that $|G^{-1}(\mathbf{u})| = (q-1)^{n-i}$ for each $\mathbf{u} \in \mathcal{W}(2, i)$.

The determination of $|\mathcal{W}(2, i)|$ is the classical problem of counting ballot sequences. It is well-known (see Theorem 1.1 in [19] or [21]) that

$$|\mathcal{W}(2, i)| = \binom{n}{i} - \binom{n}{i-1},$$

hence $|\mathcal{B} \cap X_i| = |\mathcal{W}(2, i)| =$

$$\sum_{\mathbf{u} \in \mathcal{W}(2, i)} |G^{-1}(\mathbf{u})| = (q-1)^{n-i} \cdot |\mathcal{W}(2, i)| = (q-1)^{n-i} \left(\binom{n}{i} - \binom{n}{i-1} \right).$$

□

To conclude the proof of Theorem 1.4, it suffices to verify that if s is an integer, $0 \leq s \leq n/2$, then

$$|\text{SM}(I(\mathcal{U}(n, d, q))) \cap X_{\leq s}| \leq \sum_{i=0}^s (q-1)^{n-i} \left(\binom{n}{i} - \binom{n}{i-1} \right).$$

Indeed, we have $\text{SM}(I(\mathcal{U}(n, d, q))) \subseteq \mathcal{B}$ by Corollary 3.3, hence

$$\text{SM}(I(\mathcal{U}(n, d, q))) \cap X_{\leq s} \subseteq \mathcal{B} \cap X_{\leq s}.$$

Therefore it is enough to see that

$$|\mathcal{B} \cap X_{\leq s}| \leq \sum_{i=0}^s (q-1)^{n-i} \left(\binom{n}{i} - \binom{n}{i-1} \right).$$

But this follows at once from Lemma 3.5 and the disjoint union decomposition below

$$\mathcal{B} \cap X_{\leq s} = \cup_{i=0}^s (\mathcal{B} \cap X_i).$$

This concludes the proof of Theorem 1.4. □

4 Standard monomials for Hamming spheres

Our main objective here is to prove Theorem 1.5. To this end it will be useful to consider the q -ary Hamming spheres: let $0 \leq d \leq n$, and

$$\mathcal{V}(n, d, q) := \{\mathbf{v} = (v_1, \dots, v_n) \in (q)^n : |\{i \in [n] : v_i \neq 0\}| = d\}.$$

We shall first describe the standard monomials for $I(\mathcal{V}(n, d, q))$. This will extend the corresponding result of [4] to a multivalued setting.

From $\mathcal{U}(n, d, 2) = \mathcal{V}(n, d, 2)$ and Theorem 3.1 the next statement is immediate.

Corollary 4.1 *If $0 \leq s \leq \min\{d, n - d\}$, then the standard monomials of $\mathcal{V}(n, d, 2)$ of degree at most s are exactly the standard monomials of $\mathcal{V}(n, s, 2)$.* \square

By exploiting the relation $\mathcal{V}(n, d, q) = \mathcal{V}(n, d, 2)^q$ we can now explicitly describe the normal set of $I(\mathcal{V}(n, d, q))$.

Corollary 4.2 *Let $\mathbf{u} = (u_1, \dots, u_n) \in (q)^n$, and set $c := |J(\mathbf{u})|$. We have $\mathbf{x}^{\mathbf{u}} \in \text{SM}(I(\mathcal{V}(n, d, q)))$ iff the following two conditions are satisfied:*

- a) $c \leq d$ and $|Q(\mathbf{u})| \leq \min(d - c, n - d)$.*
- b) If we write $Q(\mathbf{u}) \cup Z(\mathbf{u})$ in the form $\{j_1 < \dots < j_{n-c}\}$, and if $Q(\mathbf{u}) = \{j_{m_1} < \dots < j_{m_\ell}\}$, then $m_i \geq 2i$ holds for every $1 \leq i \leq \ell$.*

Proof. We have $\mathcal{F}^q = \mathcal{V}(n, d, q)$, where $\mathcal{F} := \binom{[n]}{d}$. For $J \subseteq [n]$ we have

$$\mathcal{F}_J = \{F \subseteq [n] : |F| = d, \text{ and } F \supseteq J\},$$

hence $\mathcal{F}_J \neq \emptyset$ iff $|J| \leq d$. From Theorem 2.2 we obtain that

$$\text{SM}(I(\mathcal{V}(n, d, q))) = \{\mathbf{x}^{\mathbf{u}} : \mathbf{u} \in (q)^n, |J(\mathbf{u})| \leq d \text{ and } x_{Q(\mathbf{u})} \in \text{SM}(I(\mathcal{F}_{J(\mathbf{u})}))\}.$$

The standard monomials of $\mathcal{F}_{J(\mathbf{u})}$ are the same as the standard monomials of the family of all $d - c$ -subsets of the set $Q(\mathbf{u}) \cup Z(\mathbf{u})$. Theorem 3.1 gives now the statement. \square

The following upper bound is a consequence of the description of the normal set $\text{SM}(I(\mathcal{V}(n, d, q)))$ given in Corollary 4.2.

Lemma 4.3 *Let $0 \leq s, d \leq n$, $n \geq 3$, $q \geq 3$ be integers. Suppose that $0 \leq s + d \leq n$. Then*

$$|\text{SM}(I(\mathcal{V}(n, d, q))) \cap X_{\leq s}| \leq \binom{n}{s} \sum_{i=0}^d (q-2)^i \binom{n-s}{i}.$$

Proof. For $0 \leq i \leq d$ we set

$$\mathcal{M}_i := \{\mathbf{x}^{\mathbf{u}} \in \text{SM}(I(\mathcal{V}(n, d, q))) : |J(\mathbf{u})| = i\}.$$

From Corollary 4.2 it is easy to verify that

$$|\mathcal{M}_i| = \binom{n}{i} (q-2)^i \binom{n-i}{d-i} = \binom{n}{d} \binom{d}{i} (q-2)^i. \quad (16)$$

From Corollary 4.1 we know that if $0 \leq \ell \leq m$ and $0 \leq s \leq \min(\ell, m - \ell)$, then

$$|\{y \in \text{SM}(I(\mathcal{V}(m, \ell, 2))), \deg y \leq s\}| = \binom{m}{s}. \quad (17)$$

For $0 \leq i \leq d$ we now set

$$\mathcal{N}_i := \mathcal{M}_i \cap X_{\leq s}.$$

Using Corollary 4.2, formulae (17) and (16) we obtain that

$$|\mathcal{N}_i| = \begin{cases} \binom{n}{i} \binom{n-i}{s} (q-2)^i = \binom{n}{s} \binom{n-s}{i} (q-2)^i & \text{if } s \leq \min(d-i, n-d) \\ \binom{n}{i} \binom{n-i}{d-i} (q-2)^i = \binom{n}{d} \binom{d}{i} (q-2)^i & \text{otherwise.} \end{cases} \quad (18)$$

We have

$$\text{SM}(I(\mathcal{V}(n, d, q))) \cap X_{\leq s} = \cup_{i=0}^d \mathcal{N}_i,$$

hence it suffices to give an upper bound for $\sum_{i=0}^d |\mathcal{N}_i|$.

Claim. For $0 \leq i \leq d$ we have

$$|\mathcal{N}_i| \leq \binom{n}{s} \binom{n-s}{i} (q-2)^i.$$

Proof. First suppose that $d-i \leq \frac{n-i}{2}$. Then $\min(d-i, n-i-(d-i)) = d-i$. If $s \leq d-i$, then (18) gives that $|\mathcal{N}_i| = \binom{n}{s} \binom{n-s}{i} (q-2)^i$. But if $s > d-i$,

then using that $s \leq n - d = n - i - (d - i)$, we get $\binom{n-i}{d-i} \leq \binom{n-i}{s}$, implying that

$$|\mathcal{N}_i| = \binom{n-i}{d-i} \binom{n}{i} (q-2)^i \leq \binom{n-i}{s} \binom{n}{i} (q-2)^i = \binom{n}{s} \binom{n-s}{i} (q-2)^i.$$

Suppose now that $d - i > \frac{n-i}{2}$. Then $\min(d - i, n - i - (d - i)) = n - d$. Since $s \leq n - d$, equation (18) implies that

$$|\mathcal{N}_i| = \binom{n}{s} \binom{n-s}{i} (q-2)^i,$$

and this gives the claim. \square

We conclude that

$$\sum_{i=0}^d |\mathcal{N}_i| \leq \sum_{i=0}^d \binom{n}{s} \binom{n-s}{i} (q-2)^i = \binom{n}{s} \sum_{i=0}^d \binom{n-s}{i} (q-2)^i.$$

This finishes the proof of the Lemma. \square

We are prepared now to prove Theorem 1.5.

Proof of Theorem 1.5: As the result is known to hold for $q = 2$, we can assume that $q > 2$. Since $\mathcal{V} \subseteq \mathcal{V}(n, d, q)$, we have also

$$\text{SM}(I(\mathcal{V})) \subseteq \text{SM}(I(\mathcal{V}(n, d, q))).$$

On the other hand, \mathcal{V} does not shatter sets of size $s+1$, hence by Proposition 2.1 we obtain that

$$\text{SM}(I(\mathcal{V})) \subseteq X_{\leq s}.$$

Using Lemma 4.3 we obtain

$$\begin{aligned} |\mathcal{V}| &= |\text{SM}(I(\mathcal{V}))| \leq |\text{SM}(I(\mathcal{V}(n, d, q))) \cap X_{\leq s}| \leq \\ &\leq \binom{n}{s} \sum_{i=0}^d \binom{n-s}{i} (q-2)^i. \end{aligned}$$

This finishes the proof of the theorem. \square

5 Concluding remarks

1. Most of our results are also valid over fields other than \mathbb{Q} . We call a field \mathbb{F} *large*, if the characteristic of \mathbb{F} is 0 or at least q . If \mathbb{F} is a large field, then we can consider $(q)^n$ as a subset of \mathbb{F}^n in a natural way. The statements in Sections 2 and 4 and the proofs we have given there are all valid over arbitrary large fields.

2. We developed a Gröbner basis approach to study shattering in a multi-valued setting. We remark here that the main result of Alon [1] also has a quite natural and simple proof in the framework of standard monomials.

Alon's Theorem states, that for every tuple system $\mathcal{V} \subseteq (q)^n$ there exists a downward closed tuple system $\mathcal{W} \subseteq (q)^n$ such that $|\mathcal{V}| = |\mathcal{W}|$ and for every $S \subseteq [n]$ we have

$$|\{\mathbf{v}|_S : \mathbf{v} \in \mathcal{W}\}| \leq |\{\mathbf{v}|_S : \mathbf{v} \in \mathcal{V}\}|.$$

In fact, let \mathbb{F} be a large field, and \prec an arbitrary term order on the polynomial ring $\mathbb{F}[x_1, \dots, x_n]$. We have then $\mathcal{V} \subseteq (q)^n \subseteq \mathbb{F}^n$, and we can consider the set of standard monomials $\text{SM}(I(\mathcal{V}))$. One can verify that the set of exponent vectors

$$\mathcal{W} = \{\mathbf{u} \in (q)^n : \mathbf{x}^{\mathbf{u}} \in \text{SM}(I(\mathcal{V}))\}$$

will meet the requirements of Alon's Theorem². Indeed, it is obvious that \mathcal{W} is downward closed and $|\mathcal{V}| = |\mathcal{W}|$. Also, suppose that $S \subseteq [n]$, and let

$$\mathcal{U} = \{\mathbf{u} \in \mathcal{W} : \text{supp}(\mathbf{u}) \subseteq S\}.$$

Using that \mathcal{W} is downward closed, we see that $|\{\mathbf{v}|_S : \mathbf{v} \in \mathcal{W}\}| = |\mathcal{U}|$. Finally, the set of monomials $\{\mathbf{x}^{\mathbf{v}} : \mathbf{v} \in \mathcal{U}\}$ is linearly independent on \mathcal{V} , and therefore on $\{\mathbf{v}|_S : \mathbf{v} \in \mathcal{V}\}$ as well.

3. To complement Theorem 1.4, we give here a simple lower bound for the size of a d -uniform tuple system \mathcal{V} , which does not shatter an $(s+1)$ -element set. We start with the following set of tuples (which shows that the Karpovsky–Milman Theorem is sharp):

$$\mathcal{W}(n, s, q) := \{\mathbf{u} = (u_1, \dots, u_n) \in (q)^n : |\{i : u_i = q-1\}| \leq s\}.$$

²Alon's Theorem is formulated in a slightly more general setting, where \mathcal{V} and \mathcal{W} are subsets of $(q_1) \times (q_2) \times \dots \times (q_n)$, where the q_i are positive integers. The proof outlined here can be extended without much difficulty to the more general case.

It is immediate that

$$|\mathcal{W}(n, s, q)| = \sum_{i=0}^s (q-1)^{n-i} \binom{n}{i}.$$

The union

$$\mathcal{W}(n, s, q) = \cup_{j=0}^{(q-1)n} (\mathcal{W}(n, s, q) \cap \mathcal{U}(n, j, q))$$

is disjoint. This implies the existence of a d such that $0 \leq d \leq (q-1)n$ and

$$|\mathcal{W}(n, s, q) \cap \mathcal{U}(n, d, q)| \geq \frac{\sum_{i=0}^s (q-1)^{n-i} \binom{n}{i}}{(q-1)n+1}.$$

Clearly $\mathcal{X} := \mathcal{W}(n, s, q) \cap \mathcal{U}(n, d, q)$ is d -uniform and $\text{sh}(\mathcal{X}) \subseteq \binom{[n]}{\leq s}$.

4. We can easily see that if $s > \lceil \frac{d}{q-1} \rceil$ and $\mathcal{V} \subseteq \mathcal{U}(n, d, q)$ is an arbitrary d -uniform tuple system, then $S \notin \text{Sh}(\mathcal{V})$, whenever $S \subseteq [n]$, $|S| = s$. For contradiction, suppose that there exists an $S \in \binom{[n]}{s}$ such that $S \in \text{Sh}(\mathcal{V})$. Define $\mathbf{v} = (v_1, \dots, v_n)$ as

$$v_j := \begin{cases} q-1, & \text{if } j \in S \\ 0, & \text{if } j \in [n] \setminus S. \end{cases}$$

Then

$$\sum_{i \in S} v_i = s(q-1) > \lceil \frac{d}{q-1} \rceil (q-1) \geq d.$$

From $S \in \text{Sh}(\mathcal{V})$ we have that there exists a $\mathbf{u} \in \mathcal{V}$ such that $\mathbf{u}|_S = \mathbf{v}|_S$. Then we have

$$\sum_{i \in S} u_i = \sum_{i \in S} v_i > d = \sum_{i=1}^n u_i,$$

a contradiction.

5. The bound of Theorem 1.5 is sharp in the case $n = s + d$, $d \leq n/2$, as witnessed by $\mathcal{V} := \mathcal{V}(n, d, q)$. The result is not sharp for $q > 2$ and $s + d < n$, as in this case the last inequality in the proof is strict.

We remark, that by $d \leq n - s$ we also have the simpler inequality

$$|\mathcal{V}| \leq \binom{n}{s} \sum_{i=0}^d \binom{n-s}{i} (q-2)^i \leq \binom{n}{s} \sum_{i=0}^{n-s} \binom{n-s}{i} (q-2)^i = \binom{n}{s} (q-1)^{n-s}.$$

For $q = 2$ this simpler inequality gives back essentially the Frankl–Pach bound.

6. For a recent partial improvement of the Frankl–Pach bound we refer to Mubayi and Zhao [20]. Shattering and related notions have many important applications in mathematics and computer science. The interested reader is referred to Babai and Frankl [5], Füredi and Pach [14], and Vapnik [25] for more details.

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